Differential rotation of relativistic superfluid in neutron stars.

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Abstract It is shown how to set up a mathematically elegant and fully relativistic superfluid model that can provide a realistic approximation (neglecting small anisotropies due to crust solidity, magnetic fields, et cetera, but allowing for the regions with vortex pinning) of the global structure of a rotating neutron star, in terms of just two independently moving constituents, one of which represents the differentially rotating neutron superfluid, while the other part represents the combination of all the other ingredients, including the degenerate electrons, the superfluid protons in the core, and the ions in the crust, whose electromagnetic interactions will tend to keep them locked together in a state of approximately rigid rotation. Order of magnitude estimates are provided for relevant parameters such as the resistive drag coefficient and the maximum pinning force.

1 Introduction

A considerable body of observational information about neutron behaviour under various circumstances is now available from pulsar timing measurements. It is generally recognised that many features can be understood only if it is assumed that – as predicted on theoretical grounds – a substantial part of the interior of such a star is in a superfluid state. It is evident that the observations should be interpreted as providing a direct measurement of the angular velocity, Ω say, of the solid outer crust of the star, with respect to which the magnetosphere responsible for the pulsed emission can be presumed to be rigidly corotating. However due to the superfluidity the coupling of the crust to the neutron fluid interior may be very weak, so that the latter will have a locally variable angular velocity, $\Omega_{\rm n}$, say that may differ very significantly from the angular velocity Ω of the rigidly rotating exterior.

In typical cases the outer part will be steadily slowing down, so that with the sign convention that Ω itself should be positive, $\Omega > 0$, one will observe a negative value $\dot\Omega < 0$ for the rate of change due to the angular momentum loss involved in the pulsar radiation process. In such circumstances one expects that the corresponding slow down of the weakly coupled superfluid interior will be subject to a delay, so that it will be in a state of relatively rapid rotation with $\Omega_{\rm n} > \Omega$. However there will also be less usual circumstances in which this inequality might be reversed, $\Omega_{\rm n} < \Omega$, for example during a period of spin up with $\dot\Omega > 0$ due to accretion. (It is even possible to conceive in which accretion is absent, and in which the inner part is still in a state of relatively rapidly rotation, $\Omega_{\rm n} > \Omega$, while the outer part is gradually spinning up, $\dot\Omega > 0$ due to the weak transfer of angular momentum from the interior, which may eventually become more imporant than the effect of pulsar radiation drag if the magnetosphere finally gets aligned with the rotation axis.)

The present work is concerned with quantitative evaluation of such effects, primarily as a contribution to the understanding of the long term evolution of the star. However this work will also be relevant to the more spectacular short term events known as "glitches", namely sudden angular velocity increases, of which the largest are characterised by $\Delta\Omega/\Omega \sim 10^{-6}$, that are followed by a period of continuous relaxation, as well as so called "noises" (fluctuations with $|\Delta\Omega|/\Omega \sim 10^{-9}$) [1]-[5]. Such effects provided the main motivation for much of the theoretical attention that has been directed, in the last two decades, to the dynamics of the neutron superfluid in the neutron stars [6][7].

All these irregularities of the angular velocity are superimposed on the long term "secular" variation, and are small in comparison with the absolute angular velocity of the star, but they are very significant for the understanding of the physics of pulsar interior. If the electromagnetic nature of the secular variation is reasonably well understood, the basic physical mechanism responsible for the rotational irregularities is still a matter of scientific debate. One of the most important but still controversial aspects concerns the "pinning" effect, whereby the vortex lines associated with the rotation of the superfluid are more or less strongly attached to the ionic lattice forming the solid crust, whose lower layers (at densities of about 10^{11} gm/cm^3 and upwards) are

interpenetrated by the neutron superfluid.

In the pinned regime, when the superfluid velocity relaxes by means of vortex creep, the theoretical superfluid relaxation times are found to be compatible with the observed postjump relaxation timecales [8][10]. If the pinning or localization of neutron vortices is not effective, they interact relatively weakly with the electron-phonon system either directly [11] [12] or through the excitation of the oscillatory degrees of freedom of vortex lines by the vortex-nucleus interaction [13]. Whether the pinned or the free flow state is operative in the inner crust depends on several uncertain factors like relative orientation of nuclear and neutron vortex lattices, the strength of the pinning potential, the timescale for repinning, etc.

An alternative model[14] for the dynamical coupling of the neutron star superfluid core is based on the dynamics of a neutron vortex with strong magnetic field. Because of the strong dependence of the vortex flow viscosity coefficient on the matter density, the core superfluid has a wide range of dynamical coupling times, which are consistent with the observed postjump relaxation time constants [3]. This provides the basis for a theory of nonstationary dynamics of neutron star core rotation [15][16] that – when applied to the analysis of angular velocity jumps and postjump relaxations – can explain the observational data for the first 6 glitches of Vela pulsar [3], and also gives correct values of the mean dynamical time (glitch or postglitch relaxation) for other pulsars in which glitches have been observed[15].

In so far as allowance for superfluidity is concerned, all the work mentionned above was carried out within the framework of a Newtonian framework, for which a for detailed phenomenological treatment has by now been developed[7][17][18]. However it is well known that relativistic corrections to the Newtonian theory for neutron stars are of the order of 20-30% , and so may be important in relation to the effects of rotation, which are of the same order. This is the reason why it has long been standard practice[19][20] for numerical work on the basic structure of neutron stars – for which a simple perfect fluid description suffices – to use a fully relativistic treatment. It is only due to technical difficulties that it has not yet become standard practice to use a similarly relativistic treatment for the study of more detailed effects for which a simple perfect fluid description is insufficient.

Although the essential theoretical machinery needed for a similarly relativistic treatment of the effects of the solidity of the crust was made available quite a long time ago[21][22][23], the technical complications involved in actually applying this machinery are such that its effective exploitation has only recently begun to be feasible in practice[24]. In so far as the effects of superfluidity are concerned, the situation was somewhat different, since the machinery needed for a fully relativistic treatment was not available at all. The standard formalism of irrotational perfect fluid mechanics would suffice in the zero temperature limit if no vortex lines were present[25] but this simplification is unjustifiable unless superfluid angular velocity $\Omega_{\rm n}$ is infinitesimally small compared with what typically occurs. In a realistic description the superfluid will be effectively fibrated by a dense latice of quantised vortex lines. Although a Newtonian description was already available[7][17][18], what was lacking for a relativistic descrip-

tion was an appropriate way of allowing at a macroscopic level for the local anisotropy due to the microstructure formed by the quantised vortex lines in the rotating superfluid interior. An elegant variational model of the kind required for this purpose has however been recently developped[26]. It fortunately turns out that the actual implementation of the relevant vortex fibration machinery is not quite as complicated as that of the elastic solid machinery[24] needed for treating elastic deformations of the crust.

The models introduced so far for the relativistic treatment of the isotropies due both to elastic solidity in the crust[21] and to superfluid vorticity in the interior[26] are all subject to the limitation that the solid or superfluid involved is supposed to be strictly conserved. Although it is easy to allow for deviations from a strictly conservative behaviour by allowing for resistive (thermal, electric, or more general) conductivity[27] it is not so obvious how these rather elaborate models should be modified to allow for what we refer to as "transfusion", meaning the transfer of matter (by the "neutron drip" process at densities above about 10^{11} gm/cm^3) from the solid ionic lattice to the ambient superfluid, a process that will usually occur too slowly to be important on short timescales, but that will be significant in the long term readjustment of the stellar equilibrium in order to allow for the effect of substantial angular momentum loss. Since the deviations from local isotropy due to elastic solidity or superfluid vorticity are never expected to exceed a small fraction of a percent, and those due to magnetic fields will also be fairly small, they can reasonably be neglected as a first approximation for the purposes of describing the long term evolution of the star. It will be shown below that within the framework of such an approximation, i.e. subject to the postulate that there is no intrinsic anisotropy, it is easy to set up a new kind two-fluid model that can adequately describe the transfusion process whereby matter is transferred between the crust and the neutron superfluid in a manner that will be adequate for our present purpose and that is likely to be useful for future applications.

The application for which this innovation is intended in the present work is a step in the bridging of the gap between previous work that used a fully relativistic treatment but that was based on the approximation of a perfect fluid description, and previous work that was based on a more refined description of the neutron star matter, but that used a Newtonian approximation for the large scale geometry. The present investigation does not include the more refined modifications, allowing for effects such as proton superconductivity, that have already been investigated in the Newtonian approximation[7][17][18], but whose relativistic description is left for future work.

The approach followed here will be rigourously theoretical in the sense that we shall refrain from making ad hoc parameter adjustments in order to match observational results whose correct interpretation may still be open to question.

2 Transfusive two constituent superfluid model

2.1 General principles.

Unlike the (non-transfusive) Landau type two constituent superfluid model that has been developped in recent years[28][29][30] to provide a microscopic (inter vortex) description allowing for the independent entropy current that will be present in a relativistic superfluid at finite temperature, the essentially different kind of two constituent superfluid model set up here allows for what we shall refer to as *transfusion*, meaning the transfer of material between the the distinct constituents which are not separately conserved. In a transfusive model of the type set up here, the "normal" constituent is not entirely dependent on (though it does include) entropy, so that it is present even at zero temperature: the primary role of this non-superfluid constituent is to represent the fraction of the baryonic material of the neutron star that is not included in the neutron superfluid, as well as the degenerate electron gas that will be present to neutralise the charge density resulting from the fact that some of these baryons will have the form of protons rather than neutrons. In the solid "crust" layers of a neutron star the protons will be concentrated together with a certain fraction of the neutrons in discrete nuclear type ions, which at the relatively moderate temperatures that are expected to apply will form a solid lattice. In the upper crust the "normal" constituent consisting of the ionic lattice and the degenerate electrons will include everything, but in the lower crust (at densities above about 10^{11} gm/cm^3) the crust will be interpenetrated by an independently moving neutron superfluid. What we refer to as "transfusion" occurs when compression takes place so that the ionic constituent undergoes a fusion process whereby neutrons are released in the form of newly created superfluid matter, or conversely, when relaxation of the pressure allows excess neutrons to be reabsorbed into the ions.

A more elaborate treatment would specifically allow for the expectation that the protons would form an independently conducting superfluid of their own at very high densities, whereas they will combine with some of the neutrons at intermediate densities, and with all of the neutrons at low densities, to form discrete ions which will tend to crystalise to form a possibly anisotropic lattice. What matters for our present purpose is that regardless of its detailed constitution, all this "normal" matter will in effect be strongly self coupled[34] by short range electromagnetic interactions so that it movement will be describable to a very good approximation as that of a single fluid with a well defined 4-velocity, u^μ say, the only independent motion being that of the (electromagnetically neutral) neutron superfluid with velocity ${u_{\rm n}}^\mu$ say. The latter will specify the direction of the part of the baryon current

$$ {n_{\rm n}}^\mu = n_{\rm n} {u_{\rm n}}^\mu \tag{1} $$

carried by the neutron superfluid, while the "normal" matter velocity specifies the direction of the remaining *collectively comoving* part

$$ {n_{\rm c}}^\mu = n_{\rm c} u^\mu \tag{2} $$

of the baryon current.

Under conditions of stationary circular flow round the axis of symmetry of the star, each of these currents will be separately conserved (making it feasible to use the more elaborate non-anisotropic models that are available [21][26]). However, during active phases of the stellar life, a certain amount of *interchange* of matter may take place between the two constituents due to the occurrence of *convection*: to be more explicit, there may be regions of rising and descending flow within which baryons are transferred respectively from or to the superfluid, so that only the total baryon current

$$n_{\rm b}{}^\mu = n_{\rm n}{}^\mu + n_{\rm c}{}^\mu \tag{3}$$

is conserved,

$$\nabla_{\!\mu} n_{\rm b}{}^\mu = 0\,, \tag{4}$$

while the separate divergence contributions $\nabla_{\!\mu} n_{\rm n}{}^\mu$ and $\nabla_{\!\mu} n_{\rm c}{}^\mu$ can be non-zero.

At densities below the "neutron drip" transition at about 10^{11} gm/cm^3, the "normal" collectively comoving constituent $n_{\rm c}{}^\mu$ will of course be identifiable with the total, $n_{\rm b}{}^\mu$. The reason why the remaining free neutron part $n_{\rm n}{}^\mu$ – which will always be present at higher densities – is presumed to be in a state of superfluidity is that the relevant condensation temperature, below which the neutrons form bosonic condensate of Cooper type pairs is estimated [32] to be at least of the order of 10^9 K, while it is expected that a newly formed neutron star will drop substantially below this temperature within a few hundred months[33]. At such comparatively low temperatures the corresponding entropy current s^μ say will not play a very important dynamical role, but for the sake of exact internal consistence it will be allowed for in the model set up here, in which it will be taken for granted that it forms part of the "normal" collectively comoving constituent so that it will have the form

$$s^\mu = s u^\mu\,. \tag{5}$$

Under conditions of sufficiently slow convection, the transfer needs not involve significant dissipation, so the process should be describable by a Lagrangian scalar, Λ say, that will depend just on the currents introduced above, of which the independent components are given just by the vectors $n_{\rm c}{}^\mu$ and $n_{\rm n}{}^\mu$ and the scalar s. Except at the highest densities, at which the distinct ions cease to exist, it would probably be a good approximation to suppose that the Lagrangian separates in the form $\Lambda = -\rho_{\rm c} - \rho_{\rm n}$ in which $\rho_{\rm c}$ is an energy density depending only on s and $n_{\rm c}$, while $\rho_{\rm n}$ is an another energy density depending only on $n_{\rm n}$, but we shall not invoke such a postulate here, i.e. we allow for the likelihood that, particularly at high densities, beyond about 10^{13} gm/cm^3, the properties of "normal" constituent will be affected by the presence of the superfluid constituent and vice versa, which means that there will be an *entrainment* effect[31][7] [34][35], whereby for example the velocity of the superfluid neutron current will no longer be parallel to the corresponding momentum. (As an alternative to the more suitable term "entrainment" this mechanism is sometimes referred to in the litterature as "drag", which is misleading because entrainment is a purely conservative, entirely

non-dissipative effect, whereas the usual kinds of drag in physics, and in particular the kind of drag to be discussed below, are essentially dissipative processes.)

If we adoped the (gas type) description embodied in the separation ansatz we would have two separate variation laws which in a fixed background would take the form $\delta\rho_{\rm c} = \Theta\delta s + \chi\delta n_{\rm c}$ and $\delta\rho_{\rm n} = \mu\delta n_{\rm n}$, in which Θ would be interpretable as the temperature, χ would be interpretable as the relativistic chemical potential per baryon in the "normal" part, and μ would be interpretable as the relativistic chemical potential per baryon (in other words the effective mass per neutron) in the superfluid part (which would be equal to its analogue in the "normal" part, i.e. $\mu = \chi$, in the particular case of a state of static thermodynamic equilibrium.)

In the less specialised (liquid type) description to be used here, there will just be a single "conglomerated" variation law, whose most general form, including allowance for a conceivable variation of the background metric, will be expressible as

$$\delta\Lambda = -\Theta\delta s + \chi_\nu\delta n_{\rm c}{}^\mu + \mu_\nu\delta n_{\rm n}{}^\nu + \tfrac{1}{2}(n_{\rm c}{}^\mu\chi^\nu + n_{\rm n}{}^\mu\mu^\mu)\delta g_{\mu\nu}\,, \qquad (6)$$

where Θ is to be interpreted as the temperature and where μ_μ and χ_μ are to be interpreted as the 4-momentum per baryon of the neutron superfluid and the "normal" constituent respectively.

To obtain suitable fluid type dynamical equations from a Lagrangian expressed as above just in terms of the relevant currents, the variation of the latter must be appropriately constrained in the manner[27] that was originally introduced for the case of a simple perfect fluid by Taub. The standard Taub procedure can be characterised as the requirement that the variation of the relevant current three form, which for the "normal" constituent in the present application will be

$$^*\!n_{{\rm c}\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} n_{\rm c}{}^\sigma\,, \qquad (7)$$

should be given by Lie transportation with respect to an associated, freely chosen, displacement vector field $\xi_{\rm c}{}^\mu$ say. This ansatz gives the well known result

$$\delta\,^*\!n_{{\rm c}\mu\nu\rho} = \xi_{\rm c}{}^\lambda\nabla_\lambda\,^*\!n_{{\rm c}\mu\nu\rho} + 3\,^*\!n_{{\rm c}\lambda[\mu\nu}\nabla_{\rho]}\xi_{\rm c}{}^\lambda\,. \qquad (8)$$

Although a variation $\delta g_{\mu\nu}$ of the metric has no effect on the fundamental current three form, $^*\!n_{{\rm c}\mu\nu\rho}$, it will contribute to the variation of the corresponding vector,

$$n_{\rm c}{}^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}\,^*\!n_{{\rm c}\nu\rho\sigma}\,, \qquad (9)$$

for which one obtains

$$\delta n_{\rm c}{}^\mu = \xi_{\rm c}{}^\nu\nabla_\nu n_{\rm c}{}^\mu - n_{\rm c}{}^\nu\nabla_\nu\xi_{\rm c}{}^\mu + n_{\rm c}{}^\mu(\nabla_\nu\xi_{\rm c}{}^\nu - \tfrac{1}{2}\gamma^{\nu\rho}\delta g_{\nu\rho}) \qquad (10)$$

in terms of the orthogonally projected metric,

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \qquad (11)$$

The corresponding variation of the unit flow vector will be given by

$$\delta u^\mu = \gamma^\mu{}_\rho ({\xi_{\rm c}}^\nu \nabla_\nu u^\rho - u^\nu \nabla_\nu {\xi_{\rm c}}^\rho) - \tfrac{1}{2} u^\mu u^\nu u^\rho \delta g_{\nu\rho} \, , \tag{12}$$

and the corresponding variation in the current amplitude $n_{\rm c}$ will be

$$\delta n_{\rm c} = \nabla_\nu (n_{\rm c} {\xi_{\rm c}}^\nu) + n_{\rm c} (u^\mu u^\nu \nabla_\mu \xi_{{\rm c}\nu} - \tfrac{1}{2} \gamma^{\mu\nu} \delta g_{\mu\nu}) \, . \tag{13}$$

Since the entropy flux is to be considered as comoving with the "normal" constituent, it is subject to a variation given by the same displacement vector $\xi_{\rm c}$, which thus gives

$$\delta s = \nabla_\nu (s {\xi_{\rm c}}^\nu) + s (u^\mu u^\nu \nabla_\mu \xi_{{\rm c}\nu} - \tfrac{1}{2} \gamma^{\mu\nu} \delta g_{\mu\nu}) \, . \tag{14}$$

On the other hand for the superfluid constituent there will be an independent displacement vector field ${\xi_{\rm n}}^\mu$ say, in terms of which the analogously constructed variation will be

$$\delta {n_{\rm n}}^\mu = {\xi_{\rm n}}^\nu \nabla_\nu {n_{\rm n}}^\mu - {n_{\rm n}}^\nu \nabla_\nu {\xi_{\rm n}}^\mu + {n_{\rm n}}^\mu (\nabla_\nu {\xi_{\rm n}}^\nu - \tfrac{1}{2} \gamma^{\nu\rho} \delta g_{\nu\rho}) \, . \tag{15}$$

The effect of this variation process on the Lagrangian density $\Vert g \Vert^{1/2} \Lambda$ itself can be seen to be expressible in the standard form

$$\Vert g \Vert^{-1/2} \delta (\Vert g \Vert^{1/2} \Lambda) = {\xi_{\rm c}}^\nu f_{{\rm c}\nu} + {\xi_{\rm n}}^\nu f_{{\rm n}\nu} + \tfrac{1}{2} \overline{T}^{\mu\nu} \delta g_{\mu\nu} + \nabla_\mu \mathcal{R}^\mu \, , \tag{16}$$

in which $f_{{\rm c}\nu}$ will be interpretable as the force density acting on the "normal" constituent, $f_{{\rm n}\nu}$ will be interpretable as the force density acting on the superfluid constituent, $\overline{T}^{\mu\nu}$ will be interpretable as the stress momentum energy density of the two constituent as a whole. The residual current $\mathcal{R}^\mu$ in the divergence will be of no importance for our present purpose (by Green's theorem it just gives a surface contribution that will vanish by the variational boundary conditions) but it is to be noted for the record that it will have the form

$$\mathcal{R}^\mu = 2 {\xi_{\rm c}}^{[\mu} u^{\nu]} (s \Theta u_\nu + n_{\rm c} \chi_\nu) + 2 {\xi_{\rm n}}^{[\mu} {n_{\rm n}}^{\nu]} \mu_\nu \, . \tag{17}$$

The conglomerated stress momentum energy density tensor can easily be read out as

$$\overline{T}^\mu{}_\nu = \Psi g^\mu_\nu + s \Theta u^\mu u_\nu + {n_{\rm c}}^\mu \chi_\nu + {n_{\rm n}}^\mu \mu_\nu \tag{18}$$

where

$$\Psi = \Lambda + s\Theta - {n_{\rm c}}^\nu \chi_\nu - {n_{\rm n}}^\nu \mu_\nu \, . \tag{19}$$

(Although this expression is not manifestly symmetric, the asymmetric contributions will automatically cancel due to the identity ${n_{\rm c}}^{[\mu} \chi^{\nu]} = \mu^{[\mu} {n_{\rm n}}^{\nu]}$). What matters most for our present purpose is the form of the respective force densities: the force law (i.e. the relevant relativistic generalisation of Newton's "second" law of motion) for the "normal" constituent is found to take the form

$$f_{{\rm c}\nu} = 2 s^\mu \nabla_{[\mu} (\Theta u_{\nu]}) + 2 {n_{\rm c}}^\mu \nabla_{[\mu} \chi_{\nu]} + \Theta u_\nu \nabla_\mu s^\mu + \chi_\nu \nabla_\mu {n_{\rm c}}^\mu \, , \tag{20}$$

while the force law for the superfluid component is found to take the simpler form

$$f_{\mathrm{n}\nu} = n_{\mathrm{n}}{}^{\mu} w_{\mu\nu} + \mu_\nu \nabla_{\!\mu} n_{\mathrm{n}}{}^{\mu} \,, \tag{21}$$

using the notation

$$w_{\mu\nu} = 2\nabla_{[\mu}\mu_{\nu]} \tag{22}$$

for the vorticity 2-form of the superfluid.

2.2 Non-dissipative "free" and "pinned" limit models.

Up to this point we have been dealing only with purely kinematical relationships, without making any physical assumptions about the form of the dynamical equations of motion. If we were to postulate that the latter were given simply by application of the variation principle for freely chosen displacement fields $\xi_{\mathrm{c}}{}^{\mu}$ and $\xi_{\mathrm{n}}{}^{\mu}$ we would obtain dynamical equations given just by the condition that the force densities $f_{\mathrm{c}\nu}$ and $f_{\mathrm{n}\nu}$ should each vanish separately, a postulate that is too restrictive for our present purpose since it would entail the separate conservation of $n_{\mathrm{c}}{}^{\mu}$ and $n_{\mathrm{n}}{}^{\mu}$, which would not be realistic for scenarios involving convection.

Before describing the more appropriate force ansatz that will be adopted below, it is to be remarked that the forces can not be specified in an entirely independent manner, in view of the action-reaction identity (the relativistic generalisation of Newton's "third" law) that is derivable from the consideration that the system will evidently be globally unaffected if both currents undergo *the same* displacement, $\xi_{\mathrm{n}}{}^{\mu} = \xi_{\mathrm{c}}{}^{\mu}$ provided that the metric itself is subject to the corresponding gauge adjustment, namely $\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\mathrm{c}\nu)}$. It can be seen from the basic variation identity (14) that since the action must be invariant with respect to any variation of this trivial kind (which merely represents an infinitesimal coordinate transformation) the forces must be subject to an identity of the form

$$f_{\mathrm{c}\nu} + f_{\mathrm{n}\nu} = \overline{f}_{\nu} \tag{23}$$

where $\overline{f}_{\nu}$ is the conglomerated *external force density* that is defined by

$$\overline{f}_{\nu} = \nabla_{\!\mu} \overline{T}^{\mu}{}_{\nu} \ . \tag{24}$$

For a system that is isolated in the strictest sense the external force density would simply vanish, but for the application to be considered here it will be necessary to take account of the action of a *non zero* external force density $\overline{f}_{\mu}$ on the star, in order to allow for the backreaction (and in particular the angular momentum loss) due to radiation by the outer magnetosphere.

Before including allowance for this and other potentially dissipative effects, it is worthwhile to present the simplest relevant model, in which interchange is appropriately allowed for in a conservative manner so that one has $\nabla_{\!\mu} s^{\mu} = 0$ even though $\nabla_{\!\mu} n_{\mathrm{c}}{}^{\mu} = -\nabla_{\!\mu} n_{\mathrm{n}}{}^{\mu} \neq 0$. The obvious way to obtain the requisite model within the

present framework is as follows. To start with it is of course necessary in this particular case to suppose that there is no external force acting on the star, i.e.

$$\overline{f}_\mu = 0 \,. \tag{25}$$

Whereas the preceeding equations in this section have been kinematic identities of a mathematically obligatory nature, (25) is the first case of a physical assumption of the kind that can be, and later on will be, relaxed in a more general treatment. The assumption (25) evidently entails by (23) that the force densities acting between the two constituents will have to be equal and opposite, i.e.

$$f_{\mathrm{n}\mu} = -f_{\mathrm{c}\mu} \,, \tag{26}$$

so it suffices to choose the physical prescription for just one of them in order to specify the other and thus to fully determine the dynamical evolution. Although the complete expression (20) is not so simple, it is to be observed that the time component in the "normal" rest frame (representing the rate of working on the "normal" constituent) as obtained by contraction with the relevant unit vector u^ν has the comparitively simple form

$$u^\nu f_{\mathrm{c}\nu} = u^\nu \chi_\nu \nabla_\mu n_{\mathrm{c}}{}^\mu - \Theta \nabla_\mu s^\mu \,. \tag{27}$$

In view of the total baryon current conservation law (3) this will be consistent with entropy conservation

$$\nabla_\mu s^\mu = 0 \tag{28}$$

if and only if the ansatz for the force $f_{\mathrm{n}\mu}$ acting on the superconducting constituent is such that the condition

$$u^\nu f_{\mathrm{n}\nu} = u^\nu \chi_\nu \nabla_\mu n_{\mathrm{n}}{}^\mu \tag{29}$$

is satisfied. This requirement is expressible in the form

$$n_{\mathrm{n}}{}^\mu w_{\mu\nu} u^\nu = u^\nu (\chi_\nu - \mu_\nu) \nabla_\mu n_{\mathrm{n}}{}^\mu \,, \tag{30}$$

in which the right hand side would obviously vanish for a non-transfusive model, as characterised by the separate conservation law

$$\nabla_\mu n_{\mathrm{n}}{}^\mu = 0 \ . \tag{31}$$

The right hand side of (30) will also vanish for a transfusive model of the kind more relevant to the neutron star applications under consideration here, in which – except for phenomena with timescales so very short as to be comparable with those of the weak interactions involved in the creation of protons from neutrons – it can be taken as a very good approximation that the condition

$$u^\nu (\chi_\nu - \mu_\nu) = 0 \,, \tag{32}$$

expressing transfusive ("chemical" type) equilibrium between the superfluid and "normal" constituents with respect to the "normal" rest frame, will be satisfied instead.

Whichever of the alternatives (31) and (32) is used, the requirement that the complete system of equations of motion be consistent with the entropy conservation condition (28) will simply reduce to the condition that the remaining dynamical equations should be such as to ensure that left hand side of (30) will vanish. There are just two obvious ways of achieving this requirement. The first way is to postulate that the superfluid momentum transport equations should be formally the same as in an ordinary single constituent barotropic perfect fluid, meaning that they should be given by the familiar expression

$$n_{\rm n}{}^{\mu} w_{\mu\nu} = 0\,. \tag{33}$$

This "free limit" equation of motion is interpretable as the condition of conservation of the superfluid vorticity flux across any two dimensional surface that is comoving with the superfluid current $n_{\rm n}{}^{\mu}$. The other way is to postulate that the superfluid dynamical equations have the alternative form

$$u^{\mu} w_{\mu\nu} = 0\,, \tag{34}$$

which is interpretable as the condition of conservation of the superfluid vorticity flux across any two dimensional surface that is comoving not with respect to the superfluid but with respect to the other "normal" constituent current $n_{\rm c}{}^{\mu}$. The latter variant is the equation of motion that is appropriate in regions where vortex pinning is effective. It will be seen from the order of magnitude estimates provided in Subsection5.2 that this "pinned limit" model, i.e. the model based on the use of (32) in conjuction with (34), will also provide a very good approximation in the deep core region of the star where that resistive drag by the "normal" constituent turns out to be extremely large, so that although they are not pinned in the strictest sense the vortices will in effect be almost exactly comoving with the "normal" constituent. On the other hand it will be seen that the first of these possibilities, i.e. the "free limit" model based on the use of (32) in conjuction with (33), should provide a very good approximation in much of the lower crust region where that resistive drag exerted on the vortices by the "normal" constituent turns out to be extremely small.

2.3 Dissipative interactions.

Our purpose in the present subsection is to set up a more general model that interpolates between the non-dissipative "free" and "pinned" extremes presented in the preceeding subsection, so as to allow for dissipative interaction between the two constituents, the kind that is most important for the application discussed below being resistive drag. For this purpose we shall retain only the general framework of Subsection 2.1 but not the more specialised conditions described in Subsection 2.2. With reference to the latter, the only stage for which the superfluidity property of the neutrons is directly relevant is the vorticity transport law, for which, instead of the idealised extreme alternatives (33) and (34) we need an intermediate modification to allow for the finite resistive drag force[11] exerted by the flux of the normal constituent on the cores of the microscopic vortices.

Since in the present work we are neglecting the energy and tension of the vortices (whose – relatively small – dynamic effects have recently been the subject of analysis in their own right [26]), the form of the vorticity transport law in the conservative limit governed by (33) is, as remarked above, just the same as it would be for a constituent of the ordinary perfectly fluid but not superfluid type. The feature that the superfluidity property as such does not have any ostensible role in this model contrasts with the situation that would apply in a microscopic description, for which the relevant vorticity 2-form $w_{\mu\nu}$ would vanish. It also contrasts with the situation that applies when drag between the normal and superfluid constituents needs to be taken into account.

The way in which the superfluidity property remains important at a macroscopic level (even when the resulting anisotropy[26] is neglected) can be explained as follows. If the neutron fluid were of the ordinary non superfluid kind, the drag force contribution ${f_{\rm n_\perp}}^\mu$ say, would be aligned with (and roughly proportional to) the relative flow vector $u^\mu + u^\nu u_{{\rm n}\nu} {u_{\rm n}}^\mu$. However a resistivity force of this familiar kind is not compatible with the property of superfluidity, which does not only require that the (macroscopically averaged) vorticity should be represented by a closed 2-form $w_{\mu\nu}$ as in conservative and dissipative fluid models of a more general kind: it is also necessary in the superfluid case that the vorticity 2-form should be consistent with a microscopic description in which the vortex cores are localised on 2-dimensional string type world sheets, which means that it should satisfy the algebraic degeneracy condition

$$\varepsilon^{\mu\nu\rho\sigma} w_{\mu\nu} w_{\rho\sigma} = 0 \, . \tag{35}$$

In conjunction with the relevant integrability condition, which is just the closure property $\nabla_{[\mu} w_{\nu\rho]} = 0$ that results automatically from the construction of the vorticity according to (22) as the exterior derivative of a momentum form, the degeneracy condition (35) ensures the existence of a congruence of two dimensional worldsheets orthogonal to the vorticity 2-form, which will be expressible in terms of its scalar amplitude,

$$w = \sqrt{w_{\mu\nu} w^{\mu\nu}/2} \, , \tag{36}$$

by

$$w_{\mu\nu} = \frac{1}{2} w \, \varepsilon_{\mu\nu\rho\sigma} \mathcal{E}^{\rho\sigma} \, , \tag{37}$$

where $\mathcal{E}^{\mu\nu}$ is the antisymmetric unit bivector (as normalised by $\mathcal{E}^{\mu\nu}\mathcal{E}_{\nu\mu} = 2$) tangential to the worldsheet – which is uniquely defined modulo the orientation convention involved in the choice of sign of the spacetime measure tensor $\varepsilon_{\mu\nu\rho\sigma}$. There will be no ambiguity of sign at all in the specification of the corresponding *fundamental tensor* of the worldsheet, namely the rank-2 tangential projection tensor that is given by

$$\eta^\rho{}_\sigma = \mathcal{E}^\rho{}_\nu \mathcal{E}^\nu{}_\sigma \, , \tag{38}$$

nor in the complementary orthogonal projection tensor

$$\perp^\rho_\sigma = g^\rho{}_\sigma - \eta^\rho{}_\sigma \, , \tag{39}$$

which will also be of rank-2. The latter will be definable directly by

$$\perp^{\rho}_{\sigma} = w^{-2} w^{\rho\nu} w_{\sigma\nu} \,. \tag{40}$$

The satisfaction of the superfluid degeneracy requirement (35) will automatically hold as a consequence whenever $w_{\mu\nu}$ has a zero eigenvalue characteristic vector such as exemplified by the current vector ${n_{\rm n}}^{\mu}$ when an equation of motion of the standard form (33) or its modification (34) is satisfied. However, when the standard equation (33) is changed in the obvious way to

$${n_{\rm n}}^{\mu} w_{\mu\nu} = f_{{\rm n}_\perp \nu} \,, \tag{41}$$

by the inclusion of a drag term $f_{{\rm n}_\perp \nu}$, the superfluidity requirement (35) will be violated unless the drag force has a very special form, which will not be compatible with the usual kind of drag proportional to the relative flow vector $u^{\mu} + u^{\nu} u_{{\rm n}\nu} {u_{\rm n}}^{\mu}$. The obvious way to make sure that the force term on the right of (41) does not exclude the existence of a zero eigenvalue characteristic for $w_{\mu\nu}$ is to postulate that it should be proportional to $w_{\mu\nu} V^{\mu}$ for some vector V^{ν}. The appropriate form for this vector can be deduced from the expression for the rate of entropy that results from substitution of (41) in (21), which leads by (27) to

$$\Theta \nabla_{\mu} s^{\mu} = u^{\nu} (\mu_{\nu} - \chi_{\nu}) \nabla_{\mu} {n_{\rm n}}^{\mu} + u^{\nu} f_{{\rm n}_\perp \nu} - u^{\nu} \overline{f}_{\nu} \,, \tag{42}$$

in which, by the preceeding considerations, the drag contribution $u^{\mu} f_{{\rm n}_\perp \mu}$ will be proportional to $u^{\mu} w_{\mu\nu} V^{\mu}$. Since any such internal contribution ought to be positive definite, in order to satisfy the second law of thermodynamics, one is naturally lead to postulate that the vector V^{μ} should itself be proportional to $w_{\mu\nu} u^{\mu}$. In view of (40) it follows that the appropriate form for the drag force density on the superfluid will be given by

$${f_{{\rm n}_\perp}}^{\mu} = \mathcal{C}_{\rm r} \perp^{\mu}_{\nu} u^{\nu} \,, \tag{43}$$

for some positive resistivity coefficient $\mathcal{C}_{\rm r}$, that can be expected to increase roughly in proportion to product of the vorticity magnitude w and the baryon density $n_{\rm c}$ of the normal constituent. It is to be remarked that a treatment involving a relativistic drag force formula of this kind has been previously developped in the context of cosmic string theory by Vilenkin[36][37]. In the specific context of neutron star matter, a resistive drag formula interpretable as the Newtonian limit kind described by (42) has been obtained on the basis of detailed microscopic analysis by Jones[11], whose quantitative estimate for the coefficient $\mathcal{C}_{\rm r}$ in the lower crust region will be discussed in Subsection 5.2, the main conclusion being that it will be very small. This means that in in the lower crust region the zero drag limit, $\mathcal{C}_{\rm r} \to 0$ will provide what for many purposes will be a very good aproximation, which will described by the non-dissipative model governed by (33). On the other hand a more recent investigation[14] of conditions in the high density core of the star indicates that the corresponding value there will be very high. This means that for this deep core region so the opposite "pinned" limit, $\mathcal{C}_{\rm r} \to \infty$, will provide a very good aproximation, which will be described by the alternative non-dissipative model that is governed by (34).

The analogous problem of the appropriate form for the law governing the superfluid creation rate $\nabla_\mu n_{\rm n}{}^\mu$ is simpler because this creation rate is just a scalar. It is evident from (42) that the natural way to ensure that this creation rate will be consistent with the second law of thermodynamic is to postulate that it should be governed by a law of the form

$$\nabla_\mu n_{\rm n}{}^\mu = \Xi\, u^\nu(\mu_\nu - \chi_\nu) \tag{44}$$

for some positive coefficient Ξ. Such a law is an obviously natural generalisation of the kind of creation rate formula that is familiar in chemical physics, but we are not aware of any microscopic analysis providing an estimate of the appropriate value for Ξ. The situation is complicated by the consideration that as far as the large scale mechanics of the neutron star is concerned, the effective rate may depend not just on microscopic processes, but also, when subduction is involved, on the rather messy process whereby the crust is broken up before it ultimately dissolves. In practice however it will suffice for many purposes, including the application to be described below, to know that the effective value of the chemical rate coefficient Ξ is sufficiently high compared with the relevant timescales of long term evolution for the local chemical equilibrium condition (32) to be an adequate approximation as an alternative to the more exact relation (44), of which it represents the non-dissipative limit as $\Xi \to \infty$. It is to be remarked that the opposite limit $\Xi \to 0$ is also non-dissipative, providing a non-transfusive treatment in which the superfluid constituent is separately conserved according to (31) (as in the more familiar non-transfusive Landau type[28][29][30] of two fluid model) which will be a good approximation for treating many high frequency processes in neutron stars, but not so relevant for the long term evolution processes to be considered here.

It is apparent from (21) that the postulates (43) and (44) can be combined in the single formula

$$f_{\rm n}{}^\rho = \mathcal{C}_{\rm r} \perp^\rho_\nu u^\nu + \Xi\, \mu^\rho u^\nu(\mu_\nu - \chi_\nu)\,, \tag{45}$$

for the force density $f_{{\rm n}\nu}$ acting on the superfluid constituent that is the primary subject of interest in the application to be described below. To complete the specification of the dynamical evolution we would need an analogously explicit formula for the force density acting on the normal constituent, which, by (23) will have the form

$$f_{\rm c}{}^\rho = \overline{f}^\rho - f_{\rm n}{}^\rho\,, \tag{46}$$

in which the external contribution $\overline{f}^\rho$ still remains to be specified.

The simplest possibility is of course that in which the external force contribution $\overline{f}^\rho$ is absent, in which case the equations listed above will be sufficient as they stand to determine the dynamical evlution. However our purpose below is to consider cases in which the star is not effectively isolated but subject to an external torque that is ultimately attributable to accretion or radiation reaction. Although it provided an indispensible guide to the formulation of the explicit expression (45) for the internal contribution $f_{\rm n}{}^\rho$, the second law of thermodynamics applies only to closed systems, so it does not provide any information about the external contribution $\overline{f}^\rho$: the final term $u^\nu\overline{f}_\nu$ in (42) might have either sign depending on the nature of the external force

involved (broadly speaking one might expect it to be positive in the case of accretion but negative for radiation reaction).

In order to cover a wide range of possible scenarios, the strategy of the analysis below will be to refrain from adopting any specific ansatz for the detailed distribution of the external force density $\overline{f}^\rho$, but to suppose that the evolution of the "normal" constituent is known in advance on the basis of other considerations, so that if the value of $\overline{f}^\rho$ were actually needed it could just be read out from (46). More specifically it will be supposed in the following work that the motion of the "normal" constituent is approximately *rigid*. In view of its elastic solid structure, this will obviously be a good approximation for the crust, and in practice it will also be a very good approximation in the deeper layers, which will be tightly coupled to the crust by forces of various (particularly viscous and magnetic) kinds. A more thorough treatment would need a detailed account of such forces, whose presence will imply deviations from the perfectly fluid description used here: this would be describable in terms of adjustments $\overline{T}^{\mu\nu} \mapsto \overline{T}^{\mu\nu} + \Delta T^{\mu\nu}$ and the ${f_{\rm c}}^\rho \mapsto {f_{\rm c}}^\rho + \Delta {f_{\rm c}}^\rho$ with $\Delta {f_{\rm c}}^\rho = \nabla_\nu(\Delta T^{\nu\rho})$, but would not directly affect the superfluid force density ${f_{\rm n}}^\rho$ in which the corresponding adjustment could be neglected as a higher order correction. So long as we are not concerned with the explicit form of the external force contribution, the effect of such adjustments can be adequately taken into account using the non adjusted formulae given above subject to the understanding that the quantity $\overline{T}^{\mu\nu}$ therein is to be interpreted as an *effective* energy momentum tensor that differs from the *true* (but not so exactly known) energy tensor $\widehat{T}^{\mu\nu}$ say by some adjustment expressible by

$$\overline{T}^{\mu\nu} = \widehat{T}^{\mu\nu} - \Delta T^{\mu\nu}\,, \tag{47}$$

while similarly the quantity $\overline{f}^\rho$ therein is to be interpreted as an *effective* external force density that differs from the *true* external force density, $\widehat{f}^\rho$ say, by a corresponding adjustment of the form

$$\overline{f}^\rho = \widehat{f}^\rho - \nabla_\mu(\Delta T^{\mu\rho})\,. \tag{48}$$

If the definition of the adjustment $\Delta T^{\mu\nu}$ is extended to include all relevant radiation and/or accreting matter, then there will remain no genuinely external force contribution, so the first term $\widehat{f}^\rho$ in (48) will simply disappear, i.e. the effective force density $\overline{f}^\rho$ will be entirely attributable to the adjustment term.

3 Angular momentum distribution.

The main subject to which we wish to apply the foregoing formalism on this occasion is the evolution of the angular momentum distribution within a neutron star.

Before we can proceed it is to be remarked that in order for the useful concepts of energy or angular momentum to be well defined in the strictest sense, it is necessary that there should be a corresponding time stationnary symmetry generator $\partial/\partial t = k^\mu\partial/\partial x^\mu$ say, or an axisymmetry generator, $\partial/\partial\phi = h^\mu\partial/\partial x^\mu$ say, whose action leaves the background spacetime structure invariant. In a Newtonian treatment using a flat

background a six parameter group of such symmetries will always exist. However in a relativistic treatment using a curved spacetime background, for which the relevant symmetry property is locally expressible by the Killing condition that $\nabla_{(\mu}k_{\nu)}$ or $\nabla_{(\mu}h_{\nu)}$ should vanish, no such solution will exist in the generic case.

Fortunately, in the context of neutron stars, this problem of principle is unimportant in practice. Although deviations from flat space geometry will typically be rather large, several tens of percent, the relevant curved spacetime geometry will nevertheless be able to be considered as time independent to within a small fraction of a percent over the timescales during which dynamical processes such as glitches occur, because the masses that are set in motion by such process are very small compared with the relevant Chandrasekhar limit: it will thus be possible to choose a time translation generator k^μ such that

$$\nabla_\mu k_\nu = \nabla_{[\mu}k_{\nu]} + \epsilon_{\mu\nu} \,, \tag{49}$$

in which the symmetric part $\epsilon_{\mu\nu} = \epsilon_{(\mu\nu)}$ is sufficiently small to be neglected for most purposes: in formal language $\epsilon_{\mu\nu} = \mathcal{O}\{L^{-1}\}$ where L is a lengthscale that is extremely large compared with the radius of the star.

In so far as axisymmetry is concerned the situation can be expected to be even better: since the masses involved even in such conspicuously non axisymmetric features as a non aligned magnetic field will be extremely small, it will usually be possible to treat the underlying spacetime geometry of a neutron star as being effectively axisymmetric to a very good approximation in the treatment of phenomena occuring not just on short and medium timescales, but even over the very long timescales with which the present investigation is chiefly concerned. In view of this, we shall proceed on the basis of the supposition that there is a well defined axisymmetry generator h^μ say that is characterised as an exact solution of the Killing equation

$$\nabla_{(\mu}h_{\nu)} = 0 \,. \tag{50}$$

The presence of such a Killing vector allows us to define the angular momentum, $\overline{J}$ say, of the star at any instant by an integral of the familiar form

$$\overline{J} = \int \overline{\mathcal{J}}^\mu \, d\Sigma_\mu \,, \tag{51}$$

where the local angular momentum current is defined by

$$\overline{\mathcal{J}}^\mu = h^\nu \overline{T}^\mu{}_\nu \,, \tag{52}$$

and the integral is taken over a spacelike hypersurface Σ characterising the instant under consideration, using the convention that the normal surface element covector $d\Sigma_\mu$ in the integrand is directed towards the *past* (in order to avoid the introduction of the minus sign that would be needed for the more usual future directe orientation convention). Although more general kinds of spacelike hypersurface might be envisaged, it will be taken for granted throughout the discussion that follows that in order to be admissible for the purpose of this definition the hypersurface must itself be invariant

under the axisymmetry action, which means that with respect to the Killing vector the normal element $d\Sigma_\mu$ must satisfy the tangentiality condition

$$h^\mu \, d\Sigma_\mu = 0 \,. \tag{53}$$

When the hypersurface used to evaluate the angular momentum is subject to the action of a time translation generator $\partial/\partial t = k^\mu \partial/\partial x^\mu$ say, the corresponding rate of variation, for which we shall use the usual abbreviation

$$\dot{\overline{J}} = \frac{d\overline{J}}{dt} \,, \tag{54}$$

will be given by the identity

$$\dot{\overline{J}} = \int (\pounds_k \overline{\mathcal{J}}^\mu + \overline{\mathcal{J}}^\mu \nabla_\nu k^\nu) \, d\Sigma_\mu \,, \tag{55}$$

in which the Lie derivative is just the commutator

$$\pounds_k \overline{\mathcal{J}}^\mu = k^\nu \nabla_\nu \overline{\mathcal{J}}^\mu - \overline{\mathcal{J}}^\nu \nabla_\nu k^\mu \,, \tag{56}$$

and the divergence contribution $\nabla_\nu k^\nu$ would vanish if the vector k^μ were taken to be an exact solution of the Killing equation, as would be possible if the background were exactly stationary, an assumption which will not be needed for our present purpose.

Independently of whether or not k^μ is actually a Killing vector, the standard time variation formula (55) can be identically rewritten in the convenient form

$$\dot{\overline{J}} = \int (k^\mu \nabla_\nu \overline{\mathcal{J}}^\nu \; + 2\nabla_\nu (\overline{\mathcal{J}}^{[\mu} k^{\nu]})) \, d\Sigma_\mu \,, \tag{57}$$

in which the last term is a divergence. This means that its contribution can be converted, by the Green theorem, into a two-dimensional integral over the boundary $\mathcal{S} = \partial\Sigma$ of the spacelike hypersurface, giving the identity

$$\dot{\overline{J}} = \int k^\mu \nabla_\nu \overline{\mathcal{J}}^\nu \, d\Sigma_\mu + \oint \overline{\mathcal{J}}^{[\mu} k^{\nu]} \, d\mathcal{S}_{\mu\nu} \,, \tag{58}$$

where $d\mathcal{S}_{\mu\nu}$ is the (timelike) normal 2-surface element of the (spacelike) boundary.

In the physical application with which we are concerned here, the boundary in (58) can be taken outside the surface of the neutron star so that its contribution drops out, and it follows from the Killing equation (50) and the definition (24) of the external force density $\overline{f}_\mu$ that the divergence of the angular momentum current (52) will be given by

$$\nabla_\mu \overline{\mathcal{J}}^\mu = h^\nu \overline{f}_\nu \,. \tag{59}$$

The rate of change of the total angular momentum is thus finally obtained in the form

$$\dot{\overline{J}} = \Gamma \,, \tag{60}$$

where $\mathit{\Gamma}$ is the total external torque as given by

$$\mathit{\Gamma} = \int h^\nu \overline{f}_\nu k^\mu \, d\Sigma_\mu \, . \tag{61}$$

Subject to the restriction (53), it can be seen from the form of the expression (18) for the total energy momentum tensor that – despite the fact that we are allowing for the possibi lity that the superfluid constituent may interact strongly with the crust and the rest of the "normal" material that is dragged along with it – there will nevertheless be an unambiguous decomposition of the total angular momentum (51) as a sum of the form

$$\overline{J} = J_{\rm c} + J_{\rm n} \, , \tag{62}$$

in which the "normal" contribution $J_{\rm c}$ (due mainly to the crust) is given by

$$J_{\rm c} = \int \mathcal{J}_{\rm c}{}^\mu \, d\Sigma_\mu \, , \tag{63}$$

with

$$\mathcal{J}_{\rm c}{}^\mu = h^\nu (s\Theta u^\mu u_\nu + n_{\rm c}{}^\mu \chi_\nu) \, , \tag{64}$$

while the superfluid contribution $J_{\rm n}$ is given by

$$J_{\rm n} = \int \mathcal{J}_{\rm n}{}^\mu \, d\Sigma_\mu \, , \tag{65}$$

with

$$\mathcal{J}_{\rm n}{}^\mu = \alpha_{\rm n} n_{\rm n}{}^\mu \, , \tag{66}$$

where $\alpha_{\rm n}$ is the angular momentum per superfluid neutron as defined simply by

$$\alpha_{\rm n} = h^\nu \mu_\nu \, . \tag{67}$$

It is to be remarked that the local angular momentum current will have the form

$$\overline{\mathcal{J}}^\mu = \mathcal{J}_{\rm c}{}^\mu + \mathcal{J}_{\rm n}{}^\mu + \check{\mathcal{J}}^\mu \, , \tag{68}$$

in which there will be an extra term

$$\check{\mathcal{J}}^\mu = \Psi h^\mu \, , \tag{69}$$

that cannot be unambiguously decomposed into collectively comoving and superfluid parts except in the separable limit for which the Lagrangian introduced in (6) can itself be decomposed in the form $\Lambda = -\rho_{\rm c} - \rho_{\rm n}$, which, as already mentionned, may be an acceptable approximation at moderate densities, but is unlikely to be accurate in the deeper regions. It can be seen however that – provided (53) is respected – the mongrel term (69) will not contribute to the integrated total.

It can be seen from the definition (21) of the force density $f_{{\rm n}\mu}$ acting on the superfluid constituent that the divergence of its angular momentum contribution will be given identically by

$$\nabla_\mu \mathcal{J}_{\rm n}{}^\mu = h^\nu f_{{\rm n}\nu} + n_{\rm n}{}^\nu \pounds_h \mu_\nu \, , \tag{70}$$

in which the Lie derivative

$$\pounds_h \mu_\nu = h^\rho \nabla_\rho \mu_\nu + \mu_\rho \nabla_\nu h^\rho \tag{71}$$

will vanish provided the stellar configuration itself (and not just the background space-time as has been assumed so far) is invariant under the axisymmetry action, in which case – by the same reasonning used to obtain (60) – the rate of change of J_n will be given simply by

$$\dot{J}_\mathrm{n} = \int h^\nu f_{\mathrm{n}\nu} k^\mu \, d\Sigma_\mu \, . \tag{72}$$

It can similarly be seen from the definition (20) of the force density $f_{\mathrm{c}\mu}$ acting on the "normal" constituent that the divergence of its angular momentum contribution will be given identically by

$$\nabla_\mu \mathcal{J}_\mathrm{c}{}^\mu = h^\nu f_{\mathrm{c}\nu} + n_\mathrm{c}{}^\nu \pounds_h \chi_\nu + s^\nu \pounds_h (\Theta u_\nu) \, . \tag{73}$$

Here again the Lie derivatives involved will vanish provided the stellar configuration shares the axisymmetry property of the background, in which case the rate of change of J_c will be given by the obvious analogue of (72), namely

$$\dot{J}_\mathrm{c} = \int h^\nu f_{\mathrm{c}\nu} k^\mu \, d\Sigma_\mu \, . \tag{74}$$

4 Evolution equations.

For the purpose of keeping account of the evolution of the angular momentum distribution, a particularly handy quantity to work with is the angular momentum per superfluid neutron, α_n, that was introduced in (67), since the axisymmetry requirement to the effect that the Lie derivative (71) should vanish, is expressible as a formula giving the gradient of α_n in terms of the vorticity in the form

$$\nabla_\mu \alpha_\mathrm{n} = w_{\mu\nu} h^\nu \, . \tag{75}$$

This relation is convenient for processing the basic superfluid equation of motion, which will be given, according to (41) and (43), by

$$n_\mathrm{n}{}^\mu w_{\mu\nu} = \mathcal{C}_\mathrm{r} \perp_{\nu\sigma} u^\sigma \, , \tag{76}$$

which can be rewritten using (40) and the orthogonality property $w_{\mu\rho} \perp^\rho_{\ \nu} = w_{\mu\nu}$ in the equivalent alternative form

$$w^2 n_\mathrm{n}{}^\nu \perp^\rho_{\ \nu} = \mathcal{C}_\mathrm{r} w^{\rho\nu} u_\nu \, . \tag{77}$$

Contracting these with the axial Killing vector h^μ one obtains a pair of dynamical equation that take the forms

$$n_\mathrm{n}{}^\nu \nabla_\nu \alpha_\mathrm{n} = \mathcal{C}_\mathrm{r} h_\perp{}^\nu u_\nu \, , \tag{78}$$

and

$$w^2 n_{n\nu} h_{\perp}{}^{\nu} = -\mathcal{C}_{r} u^{\nu} \nabla_{\nu} \alpha_{n} \,, \tag{79}$$

using the abbreviation

$$h_{\perp}{}^{\mu} = \perp^{\mu}_{\nu} h^{\nu} \,, \tag{80}$$

for the vortex sheet orthogonal projection of the Killing vector.

As already mentionned, it will be a very good approximation for our present purpose to suppose that the motion of the "normal" constituent is very nearly rigid, so that its unit flow vector will be expressible in the form

$$u^{\mu} = \gamma (k^{\mu} + \Omega h^{\mu}) \,, \tag{81}$$

where γ is a Lorentz type factor allowing for gravitational and Doppler redshifts, and where Ω is a *uniform* angular velocity (representing what is actually observed in pulsars) and k^{μ} will be an approximate solution of the Killing equation, as characterised by (49). With respect to the exact Killing vector h^{μ} and the approximate Killing vector k^{μ} that is characterised by this approximate rigidity condition, the unit flow velocity $u_{n}{}^{\mu}$ of the superfluid will be expressible in the analogous form

$$u_{n}{}^{\mu} = \gamma_{n} (k^{\mu} + \Omega_{n} h^{\mu} + v_{n}^{\mu}) \,, \tag{82}$$

in which the extra term v_{n}^{μ} allows for the possiblity of a small non circular (convective) motion which can be expected to be very small under the conditions of interest here, but in which the most important difference from (81) is that the superfluid angular velocity Ω_{n} defined by (82) is not supposed to be even approximately uniform. (We have been able to avoid the need to include an analogous small convection velocity term in (82) by taking advantage of the fact that we are not supposing that k^{μ} is an exact Killing vector, which means that there is some gauge freedom in its specification: the understanding here is that this gauge freedom has been used to absorb the small convection correction that would otherwise have been needed in (81), so that this equation is interpretable not just as a statement about the approximate rigidity of the "normal" constituent but also as a gauge fixing condition for k^{μ}.)

In terms of the uniform angular velocity Ω and the variable angular velocity Ω_{n} the information contained in (78) and (79) is expressible, using the abbreviations

$$\dot{\alpha_{n}} = k^{\mu} \nabla_{\mu} \alpha_{n} \,, \qquad h_{\perp}^{2} = h_{\perp}{}^{\mu} h_{\perp\mu} \,, \tag{83}$$

by the pair of equations

$$-\, h_{\perp}^{-2} k_{\mu} h_{\perp}{}^{\mu} = \frac{\Omega_{n} + c_{r}{}^{2} \Omega + \Omega_{-}}{1 + c_{r}{}^{2}} \,, \tag{84}$$

$$w^{-1} h_{\perp}^{-2} \dot{\alpha_{n}} = \frac{c_{r} (\Omega - \Omega_{n} - \Omega_{+})}{1 + c_{r}{}^{2}} \,, \tag{85}$$

in which $c_{\rm r}$ is the dimensionless drag coefficient given by

$$c_{\rm r}=\frac{\mathcal{C}_{\rm r}\gamma}{w n_{\rm n}\gamma_{\rm n}}\,, \tag{86}$$

where the convection contributions $\Omega_{_+}$ and $\Omega_{_-}$, expressed dimensionnally as angular velocity corrections, will be sufficiently small to be neglected for most purposes, their explicit values being given by

$$\Omega_{\pm}=h_{_\perp}^{-2}v_{\rm n}{}^{\nu}(h_{_\perp\nu}\pm c_{\rm r}{}^{\mp}w^{-1}\nabla_{\!\nu}\alpha_{\rm n})\,. \tag{87}$$

The term on the left of the first expression (84) can be interpreted as the angular velocity of the vortex array: one sees that it represents a weighted mean of the angular velocities of the normal and superfluid constituents. This is not surprising because each fluid acts so as to minimise the relative velocity between the vortices and itself. There are two extreme cases: in the limit $c_{\rm r}\to 0$, when the resistive drag is very small, the vortex array "feels" only the superfluid and co-rotates with it; in the opposite limit $c_{\rm r}\to\infty$, when the drag coefficient is very high, the vortex array co-rotates with the rigidly rotating "normal" constituent.

In view of (75) the term on the left of the second expression (85) can be interpreted in a similar way as representing the non-circular "convective" component of the velocity of the vortices in a direction orthogonal both to their direction of alignment and to the axisymmetry generator h^μ, but the reason why it is of particular interest for our present purpose is that it provides the value of the quantity $k^\nu\nabla_{\!\nu}\alpha_{\rm n}$ that measures the rate at which $\alpha_{\rm n}$ (the angular momentum per superfluid neutron) changes with time. As one would have expected, it is roughly proportional to the difference $\Omega-\Omega_{\rm n}$ with a coefficient that is large only for intermediate values of κ. It is not surprising that the coefficient is small when $c_{\rm r}$ is small since in this case the superfluid hardly "feels" the "normal part" at all. The paradox that the coefficient is *also small* for very high values of $c_{\rm r}$, i.e. when the drag force is strong, can be explained as due to the fact that in this case the friction prevents the development of the transverse "convective" motion of the vortices: in such circumstances the Joukowski force due to the Magnus effect will remains orthogonal to the direction of rotation, which renders it ineffective for reducing the difference of the rotation speeds.

The quantity given by (76) does not quite constitute the entire superfluid force density $f_{{\rm n}\mu}$ that is required for evaluating the integral in (72), since it does not include the contribution in the complete expression (45) allowing for the possible creation or destruction of the superfluid material. According to (45) the required torque density will be given by

$$h^\rho f_{{\rm n}\rho}=\mathcal{C}_{\rm r}h_{_\perp}{}^{\rho}u_\rho+\Xi\,\mu_\rho h^\rho u^\nu(\mu_\nu-\chi_\nu)\,. \tag{88}$$

Evaluating this for the configuration described by (81) and (82), using the results that have just been obtained, gives the superfluid torque density in the more explicit form

$$h^\rho f_{{\rm n}\rho}=\frac{c_{\rm r}\gamma_{\rm n}n_{\rm n}wh_{_\perp}^2}{1+c_{\rm r}{}^2}(\Omega-\Omega_{\rm n}-\Omega_{_-})+\alpha_{\rm n}\Xi\,u^\nu(\mu_\nu-\chi_\nu)\,, \tag{89}$$

in which the chemical adjustment term proportional to Ξ, and the term $\delta_{\|}$ allowing for convection can be expected to be very small compared with the dominant contribution which is proportional to the local angular velocity difference $\Omega - \Omega_{\rm n}$.

The system of equations that is thus obtained is a relative generalisation of a system of the kind that is already familiar in the Newtonian equation [11]. In the slowly rotating limit, the angular momentum contributions that we have been considering will be able to be treated just as homogeneous linear combinations of the relevant angular velocity variables, while the convection terms proportional to ${v_{\rm n}}^\mu$ and the chemical adjustment terms proportional to Ξ will be able to be neglected altogether. It can be seen that in this limit the evolution of the relevant angular velocity variables, under the influence of a weak arbitrarily time dependent external torque $\mathit{\Gamma}$, will be completely determined by the equations that have just been obtained. The way this works is particularly transparent in the separable case (meaning the case in which the entrainment effect mentionned above is neglected, which is strictly true only in the crust where there are no superconducting protons) for which the Lagrangian is decomposible in the form $\Lambda = -\rho_{\rm c} - \rho_{\rm n}$ that was mentionned above, since in this case the angular momentum per superfluid neutron, $\alpha_{\rm n}$, at any position will simply be proportional to the local value of the superfluid angular velocity $\Omega_{\rm n}$ there, while the angular momentum $J_{\rm c}$ of the entire rigidly rotating "normal" constituent will be proportional just to the single uniform angular velocity Ω that is directly observable from outside. The time evolution of $\alpha_{\rm n}$ is given by (85), while the time evolution of $J_{\rm c}$ is obtainable by substituting (89) in the equation that is obtainable via (60) and (62) from (74) in the form

$$\dot{J}_{\rm c} = \mathit{\Gamma} - \int h^\nu f_{{\rm n}\nu} k^\mu \, d\mathit{\Sigma}_\mu \, . \tag{90}$$

The same system of linear equations – namely (85) and the substitution of (89) in (90) – will still be sufficient to determine the evolution of the angular velocities for a slowly rotating system in the non-separable case, the only difference being that the matrix linearly relating the angular velocity variables to their time derivatives will have more numerous off-diagonal components: in the generic case $\alpha_{\rm n}$ will no longer be proportional just to $\Omega_{\rm n}$ but to a linear combination of $\Omega_{\rm n}$ with Ω, while $J_{\rm c}$ will no longer be proportional just to Ω but to a linear combination of Ω with some appropriately weighted linear average of the distribution of $\Omega_{\rm n}$ over the star. It is to be noticed however that, because of the expected small density of superconducting protons with respect to superfluid neutrons in the core (a few percents), the terms due to the non-separability of the Lagrangian will remain small.

5 Estimation of the relevant orders of magnitude.

Before considering the effects that were left out, in order to determine the circumstances under which their neglect is justified, we shall first estimate the magnitude of the two main kinds of force, namely the Joukowski force and the friction drag force on the

vortices, that effectively governs the motion of the material of the neutron star in the treatment that has just been described.

5.1 The Joukowski-Magnus lift force

As soon as there is a relative motion between a thin elongated structure such as an aerofoil or a superfluid vortex and the ambient fluid, the Magnus effect produces a non-dissipative "lift" force that is orthogonal to the relative motion. According to the well known formula due to Joukowski, the magnitude of the lift force per unit length, in any irrotational fluid or superfluid background, will simply be proportional to the relevant momentum circulation integral. (Joukowski's theorem was originally developped for application to aerofoils in the terrestrial atmosphere, but it is sufficiently robust to remain valid even in a highly relativistic context). The Joukowski formula is particularly convenient for application in the context of superfluids, in which the relevant momentum circulation integral, $\oint \mu_\nu \, dx^\nu$, will be given in advance just by Planck's constant or a simple fractional multiple thereof. (In the context of aerofoil theory the estimation of the value of the circulation – not to mention its control, which is the secret of success in flying – is not quite so easy).

In ordinary liquid Helium, the momentum circulation integral will just be the Planck's constant $h = 2\pi\hbar$ if the current is measured in terms of entire Helium atoms, of which each contains four baryons, but if the current is measured in baryon units, whose average momentum will be a quarter of that of a whole atom, the corresponding circulation integral will be just $h/4$. In the present application the analogue of the Helium atom is a Cooper type *pair* of neutrons, which means that as we have chosen to measure the current in baryon units the relevant momentum circulation constant will be given by $\oint \mu_\nu \, dx^\nu = h/2 = \pi\hbar$. For a circle of radius r, orthogonal to a small bunch of included vortex lines, the momentum circulation can be evaluated as $\pi r^2 w$, which means that the vorticity scalar w is interpretable as representing the momentum circulation per unit area. It follows that for a superfluid constituted by neutron pairs with the current measures in baryon units the number density of vortex lines per unit area of an orthogonal section will be given by $w/\pi\hbar$.

According to the left hand side of formula (41), which represents the relativistic version of the Joukowski-Magnus force, the magnitude $f_{\rm lift}$ of the "lift" force per unit volume due to a relative flow velocity $v_{\rm nv}$ orthogonal to the vortices of the superfluid neutrons will be given in terms of their number density $n_{\rm n}$ by $f_{\rm lift} = n_{\rm n} w \, v_{\rm nv}$. The corresponding magnitude $F_{\rm lift}$ of the "lift" force per unit length on an individual vortex will therefore be given simply by

$$F_{\rm lift} = \pi\hbar n_{\rm n} \, v_{\rm nv} \,, \tag{91}$$

which, in the non relativistic limit, can be seen to be in satisfactorily perfect agreement with what is given by the classical Joukowski "lift" force formula. (Note that the factor $\pi\hbar$ would have to be raplaced by $h = 2\pi\hbar$ if one wanted to interpret $n_{\rm n}$ as the number

density of neutron pairs, not just the number density of individual neutrons as is done here.)

5.2 The resistive drag force

For a relative flow velocity $v_{\rm cv}$ of the corotating "normal" constituent orthogonally to the vortices the magnitude $f_{\rm drag}$ of the resistive drag force per unit volume will be given according to our formula (43) by $f_{{\rm n}_\perp} = \mathcal{C}_{\rm r} v_{\rm cv}$. Dividing this by the vortex number density per unit area, $w/\pi\hbar$, as before, one sees that the corresponding expression for the force per unit length $F_{\rm drag}$ on an individual vortex line will be given by

$$F_{\rm drag} = \frac{\pi\hbar\mathcal{C}_{\rm r}}{w} v_{\rm cv} \simeq \pi\hbar n_{\rm n} c_{\rm r}\, v_{\rm cv}\,, \tag{92}$$

where $c_{\rm r}$ is the dimensionless resistivity coefficient introduced in (86).

In the context with which we are concerned, the force by which this resistive drag is balanced will be mainly provided by the Joukowski lift due to the Magnus effect. Comparing (91) and (92) it can be seen that under such conditions, the magnitude ratio of the mutually orthogonal relative velocities of the corotating "normal" matter and the superfluid with respect to the vortices will be given by

$$\frac{v_{\rm nv}}{v_{\rm cv}} = \frac{\mathcal{C}_{\rm r}}{w n_{\rm n}} \simeq c_{\rm r}\,, \tag{93}$$

from which it can be seen that $c_{\rm r}$ is interpretable as the relativistic generalisation of the drag to lift ratio that is the tangent of what is known in classical aviation theory as the "gliding angle".

Using the rough order of magnitude estimates $w \approx 2m_{\rm n}\Omega_{\rm n}$, for the vorticity, and $\alpha_{\rm n} \approx m_{\rm n}\Omega_{\rm n}\varpi^2$ for the angular momentum per superfluid neutron, where ϖ is a cylindrical radial coefficient, in terms of which we shall also have $h_\perp \approx \varpi$, it can be seen that the equation (85) provides a rough estimate for the rate of change of the local superfluid angular velocity in the form

$$\frac{\dot\Omega_{\rm n}}{\Omega_{\rm n}} \approx \frac{2c_{\rm r}(\Omega-\Omega_{\rm n})}{1+{c_{\rm r}}^2}\,, \tag{94}$$

which is interpretable as meaning that the superfluid response timescale, τ, to a change of the crust angular velocity, i.e. the characteristic lifetime for survival of a local angular velocity deviation $\Omega_{\rm n}-\Omega$ against resistive damping, will be roughly given by

$$\tau \approx \tfrac{1}{2}(c_{\rm r} + {c_{\rm r}}^{-1})|\Omega-\Omega_{\rm n}|^{-1}\,. \tag{95}$$

The application of this formula requires knowledge of the local value of just a single parameter, namely the drag ratio $c_{\rm r}$.

It can be seen that the timescale τ is shortest when $c_{\rm r}$ is of the order of unity, in which case the drag force will be of the same magnitude as the Joukowski force given

by (91). However it is likely that τ will greatly exceed the lower limit $|\Omega - \Omega_{\rm n}|^{-1}$, not only in the lower crust where one expects[11] $c_{\rm r}$ to be small, and also in the inner core where recent investigations[14] suggest that $c_{\rm r}$ is likely to be very large compared with unity.

In the lower crust region, to which the most detailed studies have been devoted, an estimate of the relevant drag coefficient has been provided by the work of Jones[11], who predicts that it should be proportional to the inverse fifth power of the relevant pairing correlation length $\xi_{\rm n}$ say, which roughly characterises the radius of the vortex cores. The value of this quantity is a sensitive function of density, with a dependence that is still subject to a considerable degree of theoretical uncertainty. For the lower crust region that is most important for the kind of application under consideration here, typical estimates[42] are in the range $\xi_{\rm n} \gtrsim 10^3 k_{_{\rm n}}^{-1}$ where $k_{_{\rm n}}$ is the Fermi wave number of the superfluid neutrons, which is related to their number density by $k_{_{\rm n}}^3 = 3\pi^2 n_{\rm n}$. Another way of expressing this is to say that the corresponding pairing energy gap, $\Delta_{_{\rm P}} \simeq \hbar^2 k_{_{\rm n}}/m_{\rm n}\xi_{\rm n}$, is in the range $\Delta_{_{\rm P}} \lesssim 10^{-3} E_{_{\rm n}}$, where $m_{\rm n}$ is the neutron mass and $E_{_{\rm n}} \simeq \hbar^2 k_{_{\rm n}}^2/2m_{\rm n}$ is the Fermi energy of the neutron superfluid.

The Jones formula is expressible as the statement that, in the lower crust region, the drag ratio $c_{\rm r}$ will be given by

$$c_{\rm r} \simeq \frac{3a_{\rm c} {E_{_{\rm P}}}^2 {\xi_{\rm n}}^{-3}}{32\pi^{3/2}\hbar N_{\rm c} m_{\rm n} n_{\rm n} {c_{\rm s}}^3}\,, \tag{96}$$

where $a_{\rm c}$ is the lattice spacing lengthscale, characterising the mean separation between the ionic crust nuclei, in which the non superfluid baryons are concentrated, and $N_{\rm c} \approx n_{\rm c} {a_{\rm c}}^3$ is the number of baryons (neutrons and protons) per nucleus, while $c_{\rm s}$ is the phonon speed in the neutron superfluid and finally $E_{_{\rm P}}$ is the "pinning" energy by which any crust nucleus located within a vortex core is bound. According to standard results developed by Alpar, Anderson, Pines and Shaham [44][45], and summarised by Ruderman[43], the latter will be given by

$$E_{_{\rm P}} \simeq \frac{\hbar^2 {a_{\rm N}}^2 n_{\rm n}}{\pi m_{\rm n} \xi_{\rm n}}\,, \tag{97}$$

where $a_{\rm N}$ is the radius of the crust nucleus, which will be given roughly by $a_{\rm N} \approx 10\,\hbar {N_{\rm c}}^{1/3}/m_{\rm n}$. Combining (96) and (97), the Jones formula is obtained in the form

$$c_{\rm r} \simeq \frac{3\hbar^3 a_{\rm c} {a_{\rm N}}^4 n_{\rm n} {\xi_{\rm n}}^{-5}}{32\pi^{7/2} N_{\rm c} m_{\rm n}^3 {c_{\rm s}}^3}\,. \tag{98}$$

It is not easy to draw precise quantitative conclusions from this formula because of the high degree of theoretical uncertainty about the density sensitive correlation length $\xi_{\rm n}$ which comes in at an inverse fifth power, but since the other lengthscales involved will presumably be smaller than or – in the case of the internuclear spacing $a_{\rm c}$ – at most comparable with $\xi_{\rm n}$, and since the phonon speed will be quite high, it seems clear that the outcome will always be small, and that it will typically be very small, $c_{\rm r} \ll 1$.

The state of affairs in the inner core is very different. According to a detailed investigation that has recently been carried out[14], the main resistive drag force in the protonically superconducting superfluid below the crust is due to the non-separability property, and the consequent entrainment, which results in the trapping of large numbers of protonic vortices by each neutron superfluid vortex: the estimated value of the resistivity due to electron scattering by these very tiny magnetised flux tubes is expressible by the formula

$$\mathcal{C}_{\rm r} = \frac{\pi\hbar\, k_{\rm e}^2\, {\delta_{\rm p}}^{3-|\alpha|}\, {\xi_{\rm p}}^{|\alpha|}}{\sqrt{3}\, 2^6\, |\alpha|}\,, \tag{99}$$

in which $k_{\rm e}$ is the Fermi wave number of the degenerate electrons, $\xi_{\rm p}$ is the pairing coherence length of the superconducting protons, $\delta_{\rm p}$ is the magnetic penetration lengthgscale, which will be given very roughly by ${\delta_{\rm p}}^2 \approx m_{\rm n} c^2/4\pi e^2 n_{\rm c}$ where e is the proton charge, and finally the index $|\alpha|$ is given by $\alpha \simeq m_{\rm p}/\Delta m_{\rm p}$ where $\Delta m_{\rm p}$ is the deviation (due to the entrainment) of the effective mass of the proton from its usual value $m_{\rm p}$, for which the numerical value is thought[35] to be somewhere in the range $-5 \lesssim \alpha \lesssim -2$. As in the previous example, the uncertainty in the relevant correlation lenghtscale, in this case $\xi_{\rm p}$, which comes in at a high and itself uncertain power, makes it hard to draw precise quantititative conclusions from (99). Nevertheless, since it involves only quantities at microscopic nuclear physical scales, and since division not just by the (nuclear order) number density but also by the macroscopic vorticity w (which will be very small by nuclear standards) is required to obtain corresponding dimensionless drag ratio, $c_{\rm r}$ it is clear that the latter will always turn out to be extremely large, $c_{\rm r} \gg 1$.

6 Range of validity of the analysis.

Before concluding we shall try to form an idea of the range of conditions under which the treatment that has just been developped should be valid, at least as a first order approximation. We shall do this by estimating the relative orders of magnitude of potentially important effects that have not been taken into account as compared with those that are treated as dominant in the analysis above.

6.1 The tension force

Among the potentially significant effects that were *not* taken into account in the treatment provided here, that first to which we shall address our attention is the deviation of the stress momentum energy contribution of the superfluid from a perfect fluid form due to the effective tension, T say, arising from the string-like nature of the vortices. An elegant relativistic formalism for the description of this effect has recently been made available [26] but it turns out not to be needed for the analysis of the very large scale long term evolution that is considered here. Although the effect of the vortex tension may in certain circumstances become important at a local level, it tends to be negligible for large scale phenomena because the associated force per unit length depends

on the bending of the vortex lines, and is proportional to the magnitude $|K|$ of the curvature vector K_ρ of the string worldsheet. For the large scale effects with which we are concerned here the relevant curvature radius $R_{\rm c} = |K|^{-1}$ can typically be expected to be of the order of the thickness of the layers involved, and thus comparable with a not insignificant fraction of the radius of the star as a whole. The tension force per unit length $F_{\rm tens}$ on an individual vortex has been shown[26] to be expressible in the form

$$F_{{\rm tens}\,\rho} = TK_\rho - \perp^{\sigma}_{\rho} \nabla_\rho T\,. \tag{100}$$

The geometric curvature vector K_ρ of the vortex worldsheet is expressible in terms of its fundamental tensor by

$$K_\rho = \eta^{\nu}{}_{\sigma}\nabla_\nu \eta^{\sigma}{}_{\rho} = \mathcal{E}^{\nu}{}_{\sigma}\nabla_\nu \mathcal{E}^{\sigma}{}_{\rho}\,. \tag{101}$$

The vortex tension T will be given[46] by the formula

$$T = \frac{\pi\hbar^2}{2}\frac{n_{\rm n}}{\mu}\ln\Big\{\frac{w_\odot}{w}\Big\}\,, \tag{102}$$

in which $w_\odot$ is a fixed parameter interpretable as the maximum order of magnitude that would be attained by the vorticity magnitude w in the limit when the vortex cores are almost in contact, so that in terms of the correlation length $\xi_{\rm n}$ introduced in Subsection 5.2, which provides an estimate of the vortex core radius, it will be given by $w_\odot \approx \hbar\xi_{\rm n}{}^{-2}$. This means that the ratio $w_\odot/w$ can be evaluated as the square of the ratio of the mean intervortex separation distance to the vortex core radius $\approx \xi_{\rm n}$, so for the rotation rates typical of neutron stars the logarithmic factor will be fairly large, $\ln\{w_\odot/w\} \approx 40$. Since this factor has only a very weak dependence on w, the gradient term in (100) will be relatively negligible. For a rough order of magnitude estimation the effective mass μ per superfluid neutron can be taken to be given by its Newtonian limit value $\mu \simeq m_{\rm n}$, so the amplitude of the tension force is found to be given by

$$F_{\rm tens} \approx \frac{20\pi\hbar^2 n_{\rm n}}{m_{\rm n} R_{\rm c}}\,. \tag{103}$$

The ratio of this tension force to the Magnus lift force $F_{\rm lift}$ given by the Joukowski formula (91) will therefore be given by

$$\frac{F_{\rm tens}}{F_{\rm lift}} \approx \frac{20\hbar}{m_{\rm n} v_{\rm nv} R_{\rm c}}\,, \tag{104}$$

where $v_{\rm nv}$ is the flow velocity of the superfluid neutrons relative to the vortices and hence, as we have seen, relative to the corotating "normal" matter in the core. For typical differential angular velocities of a few rotations per second the relative velocity will be of the order of 10^{-4} in units such that the speed of light is unity. Assuming that the relevant bending radius $R_{\rm c}$ represents a significant fraction of the stellar radius (of the order of 10^{-19} times larger than the neutron Compton radius) we see that in the

stellar core the ratio (104) will be given by $F_{\rm tens}/F_{\rm lift} \approx 10^{-15}$, which means that the tension force will indeed be entirely negligible as we have been assuming.

In contrast with the case of the core, the situation in the crust of the star is rather more delicate, since we have seen that $v_{\rm nv}$ will be much smaller than the relevant differential rotation velocity there, which will be given by $v_{\rm cv}$, i.e. by the relative flow speed of the crust material relative to the vortices. The latter determines the drag force $F_{\rm drag}$ according to the formula (92) which gives

$$\frac{F_{\rm tens}}{F_{\rm drag}} \approx \frac{20\hbar}{m_{\rm n} c_{\rm r} v_{\rm cv} R_{\rm c}} \,. \tag{105}$$

For typical differential angular velocities of a few rotations per second in the lower crust we obtain a numerical estimate of the form $F_{\rm tens}/F_{\rm drag} \approx 10^{-15} c_{\rm r}{}^{-1}$. In order to obtain the strong inequality, $F_{\rm tens} \ll F_{\rm drag}$, that we want in order to justify our neglect of the tension force in the crust as well as in the core, all we need is to be sure that although it is very small compared with unity, the drag ratio $c_{\rm r}$ will nevertheless satisfy $c_{\rm r} \gg 10^{-15}$. This does however seem a fairly safe conclusion to draw from the Jones formula (96), despite the considerable degree of uncertainty in the evaluation of the relevant microscopic parameters.

6.2 Vortex pinning

The general formalism set up in Section2 is perfectly capable of treating vortex pinning, for which it suffices to use the non-dissipative equation of motion (34) that represents the limit $c_{\rm r} \to \infty$ of the generic equation of motion (76) provided by (41) and (43). As remarked above the "pinned limit" equation of motion (34) will provide a very good approximation in the core region where we have seen that the drag ratio $c_{\rm r}$ can be expected to be extremely high. However although its macroscopic effect is similar, from a microscopic point of view the high drag effect that is predicted in the core is very different from the stricter kind of pinning whose ocurrence in the crust was originally proposed by Anderson and Itoh[47] as a mechanism that might explain the very large glitches observed in the Vela pulsar. Pinning in this strict sense is presumed to occur as a consequence of the attraction characterised by the binding energy $E_{\rm P}$ given by (26) that occurs between a superfluid vortex core and a ionic nucleus in the crust. The evaluation of the macroscopically averaged effect of this binding, both as a mechanism for static pinning and also as a source of temperature dependent resistive drag – in addition to that provided by the Jones formula (96) – has been the subject of several published discussions[44][45] [48][43] but there still seems to be considerable disagreement about the quantitative conclusions to be drawn[49]. In view of the lack of consensus among the experts, and because it does not seem to us that any of the discussions we have cited is entirely satisfactory, what we propose here is a new formula providing a lowest order approximation of the kind that seems most plausible to us.

What is generally agreed is that the magnitude $\mathcal{F}$ say of the force exerted by an ion

on a vortex core with which it is in contact will be given by

$$\mathcal{F} \approx E_{\rm P}/\xi_{\rm n} \tag{106}$$

where $\xi_{\rm n}$ is the relevant correlation length characterising the vortex core radius. However an essential point that has not always been emphasised as much as we think it ought to be is that if both the vortex cores and the ionic lattice were infinitely rigid there would be *no net pinning at all* because the diversely directed forces due to individual ions would cancel out when averaged over a sufficiently long length of vortex core. To legitimately apply the standard formula giving the pinning force $F_{\rm P}$ per unit length of vortex in the standard form[43] by the formula

$$F_{\rm P} \simeq \mathcal{F}/b_{\rm P}\,, \tag{107}$$

where $b_{\rm P}$ is the mean distance between "pinning nuclei" along the vortex, one should be very careful about how this quantity $b_{\rm P}$ is defined. In the recent discussion by Ruderman[43] it is supposed that $b_{\rm P}$ is determined by the number of ions that would happen to fall within a randomly located straight and narrow tube with radius equal to that of the vortex core which is of order $\xi_{\rm n}$, so that one simply obtains $b_{\rm P} \simeq {a_{\rm c}}^3/\pi\xi^2$. What we wish to argue however is that such a purely geometric formula is quite inappropriate, because the ionic nuclei that are counted thereon will be uniformly distributed over the vortex core so that the radially directed forces to which they give rise will entirely cancel out. The only ionic nuclei that can contribute to the net pinning effect are the extra nuclei that are relatively displaced by the local pinning force by a small distance, x say, in the range

$$\xi_{\rm n} \lesssim x \lesssim a_{\rm c} \tag{108}$$

so that they are effectively in contact with the vortex core but would have been out of range had there been no relative displacement. It is these extra nuclei that break the symmetry between forward directed and backward directed forces, because it is exclusively these extra nuclei that will be coherently positioned in such a way that their force contributions can add up constructively in the same direction. The restriction (108) is equivalent to a restriction on the magnitude of the magnitude of force per nucleus that is compatible with effective pinning: in cases for which the force $\mathcal{F}$ is too weak, so as to produce a very small average displacement, $x \lesssim \xi_{\rm n}$, there will be no net pinning because only symmetrically distributed nuclei will be involved; on the other hand if the force $\mathcal{F}$ is too strong, so that it would give $x \gtrsim a$, the crystal structure will be overridden so the vortices will be able to move through the crust as if it were fluid.

If the individual force per nucleus (106) actually is in the range compatible with (108), then the mean distance $b_{\rm P}$ between the nuclei that effectively contribute to the resulting average pinning pinning will be roughly expressible in terms of the mean displacement x by

$$b_{\rm P} \approx \frac{{a_{\rm c}}^3}{\xi_{\rm n} x}\,. \tag{109}$$

Under such conditions, there actually will be an effective pinning force per unit length, which will be given roughly by

$$F_{\rm P} \approx \frac{E_{\rm P} x}{{a_{\rm c}}^3}\,. \tag{110}$$

As well as the value of $E_{\rm P}$ and $\xi_{\rm n}$ – whose estimation as discussed in Subsection 5.2 seems to be the subject of fairly general agreement [43], at least at the level of qualitative principles, even though the results remain quantitavely rather vague – all that we still need to be able to apply the formula (110) is the appropriate value of the mean displacement x, which does not seem to have been considered in the articles we have cited, but for which it is not too difficult to make a rough guess. There are in principle two distinct ways in which the relative displacement x can be produced: either the vortex core can be bent towards the position of the attracting nucleus or the ionic nucleus can be pulled aside from its usual position in the crystal lattice.

Let us first consider the case of vortex core bending. When a segment of the length of the separation $b_{\rm P}$ between neighbouring pinning positions is subject to a small lateral displacement x, the corresponding bending angle will be of order $2x/b_{\rm P}$ and hence the total of the corresponding force that must be exerted by the relevant pinning nucleus on the vortex segments on each side will be given in order of magnitude by

$$\mathcal{F} \approx \frac{4xT}{b_{\rm P}}\,, \tag{111}$$

where T is the relevant tension. The value of T that is relevant in this case will be small compared with the formula (102) since that formula took account of all the stress within the comparitively large radius determined by the intervortex separation, whereas for our present purpose the only contribution is from the stress within a comparitively small radius. The relevant radius will be of the order of magnitude of the segment length $b_{\rm P}$, but its precise value is unimportant because it only comes in via the logarithmic factor in the relevant analogue of (102) which will have the form

$$T \simeq \frac{\pi\hbar^2 n_{\rm n}}{\mu}\ln\Big\{\frac{b_{\rm P}}{\xi_{\rm n}}\Big\}\,, \tag{112}$$

so that as a rough order of magnitude estimate, taking the logarithmic factor to be of the order of unity and using the neutron mass $m_{\rm n}$ as an estimate for the value of the relevant effective mass μ, we obtain

$$T \approx \frac{\pi^2 \xi_{\rm n} E_{\rm P}}{{a_{\rm N}}^2} \tag{113}$$

where $E_{\rm P}$ is given by the pinning energy formula (97). Equating the bending force (112) to the pinning force (106) it can thus be seen using (109) that the sustainable displacement will be given by

$$x \approx \frac{a_{\rm N}}{2\pi}\Big(\frac{a_{\rm c}}{\xi_{\rm n}}\Big)^{3/2}\,, \tag{114}$$

and hence that the separation between the relevant pinning sites will be given by

$$b_{\rm P} \approx \frac{2\pi {a_{\rm c}}^2}{a_{\rm N}}\Big(\frac{\xi_{\rm n}}{a_{\rm c}}\Big)^{1/2} . \tag{115}$$

Applying this in (107), we reach the conclusion that the pinning force per unit length obtainable from vortex core bending will be given by

$$F_{\rm P} \approx \frac{\hbar^2 {a_{\rm N}}^3 n_{\rm n}}{2\pi^2 m_{\rm n} {a_{\rm c}}^{3/2} {\xi_{\rm n}}^{5/2}} . \tag{116}$$

Let us now consider the other mechanism that can provide the relative displacement x, namely that whereby the nuclei are pulled aside from their normal positions in the ionic lattice. Since the crystal structure is due simply to electrostatic Coulomb repulsion between neighbouring ions, it is easy to see that in terms of the lattice constant $a_{\rm c}$ representing the mean separation between the ions, the restoring force $\mathcal{F}$ on an ion that is subject to a small displacement x will be given in order of magnitude by

$$\mathcal{F} \approx \frac{Z^2 e^2 x}{{a_{\rm c}}^3} \tag{117}$$

where e is the electronic charge coupling constant and Z is the ionic charge number, which is expected to be typically of order $Z \approx \frac{1}{2} \times 10^3$. Equating this to the pinning force (106) we obtain

$$x \approx \frac{{a_{\rm c}}^3 E_{\rm P}}{Z^2 e^2 \xi_{\rm n}} , \tag{118}$$

which corresponds to a separation between the relevant pinning sites given simply by

$$b_{\rm P} \approx \frac{Z^2 e^2}{E_{\rm P}} . \tag{119}$$

It follows from (107) or (110) that the pinning force per unit length obtainable by displacing ions from their lattice sites will be given by

$$F_{\rm P} \approx \frac{{E_{\rm P}}^2}{Z^2 e^2 \xi_{\rm n}} . \tag{120}$$

In view of their sensitivity to the rather vaguely known correlation length $\xi_{\rm n}$, the quantitative evaluation of the pinning forces predicted by the formulae (116) and (120) is not easy, but it is clear that the latter will be relatively negligible, i.e. the vortex bending contribution (116) will dominate, except perhaps in the deepest part of the crust where the two kinds of contribution may be comparable if $\xi_{\rm n}$ is sufficiently small. It is also clear that even the bending contribution (116) will be much too weak to resist the Joukowski-Magnus force given by (91) – i.e. the ratio

$$\frac{F_{\rm lift}}{F_{\rm P}} \approx \frac{2\pi^3 m_{\rm n} {a_{\rm c}}^{3/2} {\xi_{\rm n}}^{5/2} v_{\rm nv}}{\hbar^2 {a_{\rm N}}^3} , \tag{121}$$

will be much greater than unity – whenever the relative flow speed $v_{\rm nv}$ exceeds a depth dependent critical value corresponding to a differential rotation rate that in most parts of the crust will be a small fraction of a revolution per second.

7 Conclusions

The new transfusive kind of relativistic superfluid model developed here in Section 2 can in principle provide the framework for a rough but realistic representation of the bulk motion of the material in a neutron star. Such a representation should be useful as a first approximation that can be taken as a basis on which a more accurate description including details of diverse secondary phenomena (notably magnetic effects) can then be developed by successive approximations. However in order to carry out such a program in practice it will be necessary to obtain more complete information about the requisite equations of state and in particular the parameter dependence on the various forces involved, for which results quoted in Section 5, as well as the new formulae derived here in Section 6, should be considered just as tentative provisional estimates. These estimates may need substantial revision to allow for detailed effects that have not yet been taken into account here at all, and that have not yet been sufficiently explored in preceeding published litterature. Further work will also required for the evaluation of the dissipative transfusion rate coefficient Ξ in (45), though this is not of primary importance because it can be expected to be so high that non-dissipative transfusive equilibrium equation (32) should provide an approximation that will be more than sufficiently acurate for the purposes we have in mind (since the electro-weak interactions involved will presumably be very rapid compared with the relevant neutron star evolution timescales). An example of a potentially more important effect that has not been discussed here is thermal barrier penetration, whose relevance for pinning has been noticed in previous work[44][45][48], but for which it much still needs to be done before the results can be considered reliable.

Assuming the modifications due to such provisionally neglected thermal and other effects are not large enough to invalidate the prediction based on the Jones formula (98) of a very low range of values for the drag ratio $c_{\rm r}$ in the crust, it is to be anticipated that the resistive damping timescale τ given by (95) can be large enough compared with the pulsar slow down rate $\dot\Omega$ to allow the build up of a differential rotation with an order of magnitude $\tau\dot\Omega$ that may easily exceed the critical value beyond which vortex pinning will break down, which according to the reasonning in the preceeding subsection requires a difference of a few revolutions per second at the very most. Beyond this threshold the vortex lines in the crust will tend to corotate with the neutron superfluid rather than the – in general more slowly rotating – "normal" matter. On the other hand, in the very high density layers below the crust the resistive drag is expected to be so high that the vortices will in fact be dragged along with the "normal" material in a manner that simulates the effect of pinning, but with the important difference that this strong drag effect is not subject to a threshold such as that beyond which pinning in the strict sense will break down.

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